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Duality for multiparametric quantum $GL(n)$

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Abstract. We show that the Hopf algebra $\mathcal{U}_{uq}$ dual to the multiparameter matrix quantum group $GL_{uq}(n)$ may be realized *a la* Sudbery, i.e. tangent vectors at the identity. Furthermore, we give the Cartan–Weyl basis of $\mathcal{U}_{uq}$ and show that this is consistent with the duality. We show that as a commutation algebra $\mathcal{U}_{uq} \cong U_u(sl(n, \mathbb{C})) \otimes U_u(\mathcal{Z})$, where $\mathcal{Z}$ is one-dimensional and $U_u(\mathcal{Z})$ is a central algebra in $\mathcal{U}_{uq}$. However, as a co-algebra $\mathcal{U}_{uq}$ cannot be split in this way and depends on all parameters.

1. Introduction

Quantum groups first appeared as *quantum algebras*, i.e. as one-parameter deformations of the universal enveloping algebras $U_q(\mathcal{G})$ of complex simple Lie algebras $\mathcal{G}$, in the study of the algebraic aspects of quantum integrable systems in [1–6]. Then quantum algebras related to trigonometric solutions of the quantum Yang–Baxter equation were axiomatically introduced as (pseudo) quasi-triangular Hopf algebras in [7–10]. Other approaches to quantum groups, in which the objects may be called *quantum matrix groups* and are Hopf algebras in duality to the quantum algebras, were developed in [11–15]. Later, the matrix quantum group approaches were pursued further, in particular finding consistent multiparametric deformations in [16–28].

In the present paper we find the dual algebra to the multiparameter matrix quantum group $GL_{uq}(n)$ which is the $(n(n-1)/2+1)$-parameter deformation of $GL(n)$ given in [17, 22]. We partially use the approach of [29] to postulate the pairings between the generating elements of the two algebras. We show that the algebra $\mathcal{U}_{uq}$ dual as a Hopf algebra to the multiparametric deformation $GL_{uq}(n)$ may be realized a la Sudbery [29]. Furthermore, we give the Cartan–Weyl basis of $\mathcal{U}_{uq}$ and show that this is consistent with Sudbery duality. The algebra $\mathcal{U}_{uq}$ contains a central Hopf algebra $U_u(\mathcal{Z})$, where $\mathcal{Z}$ is one-dimensional. Moreover, as a commutation algebra we have $\mathcal{U}_{uq} \cong U_u(sl(n, \mathbb{C})) \otimes U_u(\mathcal{Z})$. However, as a co-algebra $\mathcal{U}_{uq}$ cannot be split in this way and depends on all parameters.

2. Dual algebra of multiparametric deformation of $GL(n)$

In [12, 13] Manin has considered a family of quantum groups, deformations of the algebra of polynomial functions on $GL(n)$, depending on $n(n-1)/2$ parameters. Later, different

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multiparameter deformations were found in [16–28]. The maximal number of parameters for $GL(n)$ is $N = n(n-1)/2 + 1$ [17, 22]. Following [17] we denote these N parameters by u and q_{ij}, $1 \leqslant i < j \leqslant n$, and also for shortness by the pair u, q.

Let us consider an $n \times n$ quantum matrix M with non-commuting matrix elements a_{ij}, $1 \leqslant i, j \leqslant n$. The matrix quantum group $GL_{uq}(n)$ is generated by the matrix elements a_{ij} with the following commutation relations [17, 22]:

$$a_{ij}a_{il} = pa_{il}a_{ij} \qquad \text{for } j < l \tag{1a}$$

$$a_{ij}a_{kj} = qa_{kj}a_{ij} \qquad \text{for } i < k \tag{1b}$$

$$pa_{il}a_{kj} = qa_{kj}a_{il} \qquad \text{for } i < k,\ j < l \tag{1c}$$

$$uqa_{kl}a_{ij} - (up)^{-1}a_{ij}a_{kl} = \lambda a_{il}a_{kj} \qquad \text{for } i < k,\ j < l \tag{1d}$$

$$p = q_{jl}/u^2 \qquad q = 1/q_{ik} \qquad \lambda = u - 1/u. \tag{1e}$$

Considered as a bialgebra, it has the following comultiplication $\delta_{\mathcal{A}}$ and co-unit $\varepsilon_{\mathcal{A}}$:

$$\delta_{\mathcal{A}}(a_{ij}) = \sum_{k=1}^{n} a_{ik} \otimes a_{kj} \qquad \varepsilon_{\mathcal{A}}(a_{ij}) = \delta_{ij}. \tag{2}$$

This algebra has determinant $\mathcal{D}$ given by [22, 17]:

$$\begin{aligned}\mathcal{D} &= \sum_{\rho \in S_n} \epsilon(\rho) a_{1,\rho(1)} \dots a_{n,\rho(n)} \\ &= \sum_{\rho \in S_n} \epsilon'(\rho) a_{\rho(1),1} \dots a_{\rho(n),n}\end{aligned} \tag{3}$$

where summations are over all permutations ρ of $\{1, \dots, n\}$ and the quantum signatures are

$$\epsilon(\rho) = \prod_{\substack{j<k \\ \rho(j)>\rho(k)}} (-q_{\rho(k)\rho(j)}/u^2) \tag{4a}$$

$$\epsilon'(\rho) = \prod_{\substack{j<k \\ \rho(j)>\rho(k)}} (-1/q_{\rho(k)\rho(j)}). \tag{4b}$$

The determinant obeys [17, 22]:

$$\delta_{\mathcal{A}}(\mathcal{D}) = \mathcal{D} \otimes \mathcal{D} \qquad \varepsilon_{\mathcal{A}}(\mathcal{D}) = 1. \tag{5}$$

The determinant is almost central, i.e. it q-commutes with the elements a_{ij} [22]:

$$a_{ik}\mathcal{D} = \frac{\prod_{j=1}^{k-1} q_{jk}^{-1} \prod_{l=k+1}^{n} q_{kl}/u^2}{\prod_{s=1}^{i-1} q_{si}^{-1} \prod_{t=i+1}^{n} q_{it}/u^2} \mathcal{D} a_{ik}. \tag{6}$$

Further, if $\mathcal{D} \neq 0$ one extends the algebra by an element $\mathcal{D}^{-1}$ which obeys [17, 22]:

$$\mathcal{D}\mathcal{D}^{-1} = \mathcal{D}^{-1}\mathcal{D} = 1_{\mathcal{A}}. \tag{7}$$

Note that for $q_{ij} = u$ for all i, j then the element $\mathcal{D}$ is central and it is possible that $\mathcal{D} = \mathcal{D}^{-1} = 1_{\mathcal{A}}$.

Next one defines the left and right quantum cofactor matrices A_{ij} and A'_{ij} [17]:

$$A_{ij} = \sum_{\rho(i)=j} \frac{\epsilon(\rho \circ \sigma_i)}{\epsilon(\sigma_i)} a_{1,\rho(1)} \ldots \widehat{a}_{ij} \ldots a_{n,\rho(n)} \tag{8a}$$

$$A'_{ij} = \sum_{\rho(j)=i} \frac{\epsilon(\rho \circ \sigma'_i)}{\epsilon(\sigma'_i)} a_{\rho(1),1} \ldots \widehat{a}_{ij} \ldots a_{\rho(n),n} \tag{8b}$$

where σ_i and σ'_j denote the cyclic permutations:

$$\sigma_i = \{i, \ldots, 1\} \qquad \sigma'_j = \{j, \ldots, n\} \tag{9}$$

and the notation $\hat{x}$ indicates that x is to be omitted. Now one can show that [22, 17]:

$$\sum_j a_{ij} A_{kj} = \sum_j A'_{ji} a_{jk} = \mathrm{d}_{ik} \mathcal{D} \tag{10}$$

and obtain the left and right inverse [22, 17]:

$$M^{-1} = \mathcal{D}^{-1} A' = A \mathcal{D}^{-1}. \tag{11}$$

Thus, one can introduce the antipode in $GL_{uq}(n)$ [17, 22]:

$$\gamma_{\mathcal{A}}(a_{ij}) = \mathcal{D}^{-1} A'_{ji} = A_{ji} \mathcal{D}^{-1}. \tag{12}$$

We are looking for the dual algebra to $GL_{uq}(n)$. Let us recall that two bialgebras $\mathcal{U}, \mathcal{A}$ are said to be *in duality* if there exists a doubly non-degenerate bilinear form

$$\langle , \rangle : \mathcal{U} \times \mathcal{A} \to \mathbb{C} \qquad \langle , \rangle : (u, a) \mapsto \langle u, a \rangle \qquad u \in \mathcal{U},\ a \in \mathcal{A} \tag{13}$$

such that, for $u, v \in \mathcal{U}$, $a, b \in \mathcal{A}$,

$$\langle u, ab \rangle = \langle \delta_{\mathcal{U}}(u), a \otimes b \rangle \qquad \langle uv, a \rangle = \langle u \otimes v, \delta_{\mathcal{A}}(a) \rangle \tag{14}$$

$$\langle 1_{\mathcal{U}}, a \rangle = \varepsilon_{\mathcal{A}}(a) \qquad \langle u, 1_{\mathcal{A}} \rangle = \varepsilon_{\mathcal{U}}(u). \tag{15}$$

Two Hopf algebras $\mathcal{U}, \mathcal{A}$ are said to be *in duality* if they are in duality as bialgebras and if

$$\langle \gamma_{\mathcal{U}}(u), a \rangle = \langle u, \gamma_{\mathcal{A}}(a) \rangle. \tag{16}$$

It is enough to define the pairing (13) between the generating elements of the two algebras. The pairing between any other elements of $\mathcal{U}, \mathcal{A}$ then follows from relations (14), (15) and the standard bilinear form inherited by the tensor product.

However, we do not know the dual algebra completely. Then we need to know the action of the algebra $\mathcal{U}_{uq}$ dual to $GL_{uq}(n)$ on every element of $GL_{uq}(n)$. The basis of $GL_{uq}(n)$ consists of monomials

$$f = (a_{11})^{k_1} \ldots (a_{nn})^{k_n} (a_{12})^{m_{12}} \ldots (a_{n-1,n})^{m_{n-1,n}} (a_{n,n-1})^{n_{n,n-1}} \ldots (a_{21})^{n_{21}} \tag{17}$$

where $k_i, m_{ij}, n_{ij} \in \mathbb{Z}_+$ and we have used the so-called normal ordering of the elements a_{ij}. Namely, we first put the elements a_{ii}; then we put the elements a_{ij} with $i < j$ in lexicographic order, i.e. if $i < k$ then a_{ij} $(i < j)$ is before a_{kl} $(k < l)$ and a_{ti} $(t < i)$ is before a_{tk}; finally we put the elements a_{ij} with $i > j$ in antilexicographic order, i.e. if $i > k$ then a_{ij} $(i > j)$ is before a_{kl} $(k > l)$ and a_{ti} $(t > i)$ is before a_{tk}.

For the definition of the duality we shall use the approach which Sudbery invented for $GL_q(2)$ [29] and which was used in [30] for $GL_{p,q}(2)$. The dual algebra $\mathcal{U}_{uq}$ may be called the algebra of tangent vectors at the identity of $GL_{uq}(n)$, namely we define the pairing only for the monomials in the normal order (17) as follows

$$\langle D_i, f\rangle \equiv k_i\delta_{m0}\delta_{n0} \qquad 1 \leqslant i \leqslant n \tag{18a}$$

$$\langle E_{ij}, f\rangle \equiv \delta_{m_{ij}1}\delta^{ij}_{m0}\delta_{n0} \qquad 1 \leqslant i < j \leqslant n \tag{18b}$$

$$\langle F_{ij}, f\rangle \equiv \delta_{n_{ij}1}\delta_{m0}\delta^{ij}_{n0} \qquad 1 \leqslant j < i \leqslant n \tag{18c}$$

$$\langle 1_{\mathcal{U}}, f\rangle \equiv \delta_{m0}\delta_{n0} \tag{18d}$$

$$\delta_{m0} = \prod_{1\leqslant j<k\leqslant n} \delta_{m_{jk}0} \qquad \delta_{n0} = \prod_{1\leqslant k<j\leqslant n} \delta_{n_{jk}0} \tag{18e}$$

$$\delta^{ij}_{m0} = \prod_{\substack{1\leqslant k<l\leqslant n\\ (k,l)\neq(i,j)}} \delta_{m_{kl}0} \qquad \delta^{ij}_{n0} = \prod_{\substack{1\leqslant l<k\leqslant n\\ (k,l)\neq(i,j)}} \delta_{n_{kl}0}. \tag{18f}$$

If some monomial is not in normal order then it should be brought to this order using commutation relations (1) and then (18) can be applied. Thus following [29] we can interpret formulae (18) as

$$\langle Y, f\rangle \equiv \varepsilon_{\mathcal{A}}\left(\frac{\partial f}{\partial y}\right) \tag{19}$$

where y is a generating element of $GL_{uq}(n)$ and differentiation is from the right. Note also that from (18) it follows that

$$\langle Y, 1_{\mathcal{A}}\rangle = 0 \qquad Y = D_i, E_{ij}, F_{ij}. \tag{20}$$

3. Commutation relations of the dual algebra

To obtain the the commutation relations between the generators D_i, E_{ij}, F_{ij} we first need to evaluate the action of their bilinear products on the elements of $GL_{uq}(n)$. We shall show that it is enough to do this for the Chevalley-like generators D_i, $1 \leqslant i \leqslant n$, $E_i \equiv E_{i,i+1}$, $F_i \equiv F_{i+1,i}$, $1 \leqslant i \leqslant n-1$. Then through them we shall express the rest of the generators E_{ij}, F_{ij}.

Using the defining relations and the second of relations (14) we obtain:

$$\langle D_iD_j, f\rangle = \langle D_jD_i, f\rangle = k_ik_j\delta_{m0}\delta_{n0} \qquad 1 \leqslant i, j \leqslant n \tag{21}$$

$$\begin{aligned} &\langle D_iE_j, f\rangle = (k_i + \delta_{ij})\delta_{m_{j,j+1}1}\delta^{j}_{m0}\delta_{n0} = (k_i+\delta_{ij})\langle E_j, f\rangle \qquad \delta^j_{m0} = \delta^{j,j+1}_{m0} \\ &\langle E_jD_i, f\rangle = (k_i + \delta_{i,j+1})\delta_{m_{j,j+1}1}\delta^{j}_{m0}\delta_{n0} = (k_i+\delta_{i,j+1})\langle E_j, f\rangle \end{aligned} \tag{22}$$

$$\langle D_i F_j, f\rangle = (k_i + \delta_{i,j+1})\delta_{n_{j+1,j}1}\delta_{m0}\delta^j_{n0} = (k_i + \delta_{i,j+1})\langle F_j, f\rangle \qquad \delta^j_{n0} = \delta^{j+1,j}_{n0}$$
$$\langle F_j D_i, f\rangle = (k_i + \delta_{ij})\delta_{n_{j+1,j}1}\delta_{m0}\delta^j_{n0} = (k_i + \delta_{ij})\langle F_j, f\rangle \tag{23}$$

$$\langle E_i F_i, f\rangle = u^{-2k_{i+1}}\frac{u^{2k_i}-1}{u^2-1}\delta_{m0}\delta_{n0} + q^{-1}_{i,i+1}\delta_{m_{i,i+1}1}\delta_{n_{i+1,i}1}\delta^i_{m0}\delta^i_{n0}$$
$$\langle F_i E_i, f\rangle = \frac{u^{-2k_{i+1}}-1}{u^{-2}-1}\delta_{m0}\delta_{n0} + u^2 q^{-1}_{i,i+1}\delta_{m_{i,i+1}1}\delta_{n_{i+1,i}1}\delta^i_{m0}\delta^i_{n0} \tag{24}$$

$$\langle E_i F_j, f\rangle = e_{ij}\delta_{m_{i,i+1}1}\delta_{n_{j+1,j}1}\delta^i_{m0}\delta^j_{n0} \qquad i \neq j$$
$$e_{ij} = \begin{cases} q_{i+1,j+1}/q_{i,j+1} & \text{for } i < j \\ u^2/q_{i,i+1} & \text{for } i = j+1 \\ q_{j+1,i}/q_{j+1,i+1} & \text{for } i > j+1 \end{cases}$$
$$\langle F_j E_i, f\rangle = f_{ij}\delta_{m_{i,i+1}1}\delta_{n_{j+1,j}1}\delta^i_{m0}\delta^j_{n0} \qquad i \neq j$$
$$f_{ij} = \begin{cases} q_{i+1,j}/q_{ij} & \text{for } i < j-1 \\ 1/q_{i,i+1} & \text{for } i = j-1 \\ q_{ji}/q_{j,i+1} & \text{for } i > j \end{cases} \tag{25}$$

$$\langle E_i E_{i+1}, f\rangle = q^{-1}_{i,i+1}\delta_{m_{i,i+1}1}\delta_{m_{i+1,i+2}1}\delta^{i,i+1}_{m0}\delta_{n0} + \delta_{m_{i,i+2}1}\delta^{i,i+2}_{m0}\delta_{n0}$$
$$\langle E_i E_j, f\rangle = e'_{ij}\delta_{m_{i,i+1}1}\delta_{m_{j,j+1}1}\delta^{ij}_{m0}\delta_{n0} \qquad i \neq j-1$$
$$e'_{ij} = \begin{cases} q_{i+1,j}/q_{ij} & \text{for } i < j-1 \\ (1+u^2)/q_{i,i+1} & \text{for } i = j \\ q_{j+1,i+1}/q_{j,i+1} & \text{for } i > j \end{cases} \tag{26}$$

$$\langle F_{i+1} F_i, f\rangle = q_{i+1,i+2}\delta_{n_{i+1,i}1}\delta_{n_{i+2,i+1}1}\delta^{i,i+1}_{n0}\delta_{m0} + \delta_{n_{i+2,i}1}\delta^{i+2,i}_{n0}\delta_{m0}$$
$$\langle F_i F_j, f\rangle = f'_{ij}\delta_{n_{i+1,i}1}\delta_{n_{j+1,j}1}\delta^{ij}_{n0}\delta_{m0},\ i \neq j+1$$
$$f'_{ij} = \begin{cases} q_{i,j+1}/q_{ij} & \text{for } i < j \\ (1+u^{-2})q_{i,i+1} & \text{for } i = j \\ q_{j+1,i+1}/q_{j+1,i} & \text{for } i > j+1. \end{cases} \tag{27}$$

Thus we have the following commutation relations:

$$[D_i, D_j] = 0 \tag{28a}$$
$$[D_i, E_j] = (\delta_{ij} - \delta_{i,j+1})E_j \qquad [D_i, F_j] = (-\delta_{ij} + \delta_{i,j+1})F_j \tag{28b}$$
$$uE_iF_i - u^{-1}F_iE_i = \lambda^{-1}(u^{2(D_i - D_{i+1})} - 1_{\mathcal{U}}) \tag{28c}$$
$$E_iF_j = g_{ij}F_jE_i \qquad i \neq j \tag{28d}$$
$$g_{ij} = e_{ij}/f_{ij} = \begin{cases} q_{ij}q_{i+1,j+1}/q_{i,j+1}q_{i+1,j} & \text{for } i < j-1 \\ q_{i,i+1}q_{i+1,i+2}/q_{i,i+2} & \text{for } i = j-1 \\ u^2 q_{i-1,i+1}/q_{i-1,i}q_{i,i+1} & \text{for } i = j+1 \\ q_{j,i+1}q_{j+1,i}/q_{ji}q_{j+1,i+1} & \text{for } i > j+1 \end{cases} \tag{28e}$$

$$E_iE_j = g_{ij}^{-1}E_jE_i \qquad i < j-1 \tag{28f}$$

$$F_iF_j = g_{ij}^{-1}F_jF_i \qquad i > j+1. \tag{28g}$$

It is convenient to use as well as the generators D_i the generators

$$K = D_1 + \cdots + D_n \qquad H_i = D_i - D_{i+1} \qquad 1 \leqslant i \leqslant n-1. \tag{29}$$

and we shall give many results for both sets. Let us note that the generator K commutes with all generators D_i, E_i, F_i:

$$[K, D_i] = 0 \qquad [K, E_i] = 0 \qquad [K, F_i] = 0 \tag{30}$$

while the generators H_i, E_i, F_i also form a commutation subalgebra, namely instead of formulae (28a–c) we have

$$[H_i, H_j] = 0 \tag{31a}$$

$$[H_i, E_j] = c_{ij}E_j \qquad [H_i, F_j] = -c_{ij}F_j \tag{31b}$$

$$c_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i+1,j} \qquad 1 \leqslant i, j \leqslant n-1 \tag{31c}$$

$$uE_iF_i - u^{-1}F_iE_i = \lambda^{-1}(u^{2H_i} - 1_{\mathcal{U}}) \tag{31d}$$

where the numbers c_{ij} form the Cartan matrix of the algebra $A_{n-1} = sl(n, \mathbb{C})$.

Similarly to (28) we derive the analogue of the Serre relations:

$$p_i^{\pm}E_i^2E_{i\pm1} - (u + u^{-1})E_iE_{i\pm1}E_i + (p_i^{\pm})^{-1}E_{i\pm1}E_i^2 = 0 \tag{32a}$$

$$p_i^{\pm}F_i^2F_{i\pm1} - (u + u^{-1})F_iF_{i\pm1}F_i + (p_i^{\pm})^{-1}F_{i\pm1}F_i^2 = 0 \tag{32b}$$

$$p_i^{+} = q_{i,i+1}q_{i+1,i+2}/uq_{i,i+2} \qquad p_i^{-} = uq_{i-1,i+1}/q_{i-1,i}q_{i,i+1}. \tag{32c}$$

Now we shall express the rest of the generators E_{ij}, F_{ij} through the Chevalley-like ones. First let us rewrite relations (32) for the '+' sign in a more suggestive way:

$$u^{-1}p_iE_i(E_iE_{i+1} - p_i^{-1}E_{i+1}E_i) - u(E_iE_{i+1} - p_i^{-1}E_{i+1}E_i)E_i = 0 \tag{33a}$$

$$up_i^{-1}(F_{i+1}F_i - p_iF_iF_{i+1})F_i - u^{-1}F_i(F_{i+1}F_i - p_iF_iF_{i+1}) = 0 \tag{33b}$$

$$p_i = q_{i,i+1}q_{i+1,i+2}/q_{i,i+2}. \tag{33c}$$

Thus we are prompted to define generators inductively in a way similar to that used for one-parameter deformations (cf [9, 31]):

$$\begin{aligned} E_{ij} &\equiv E_iE_{i+1,j} - p_{ij}^{-1}E_{i+1,j}E_i \qquad i < j \\ F_{ij} &\equiv F_{i,j+1}F_j - p_{ij}F_jF_{i,j+1} \qquad i > j \\ p_{ij} &= q_{i,i+1}q_{i+1,j}/q_{ij}. \end{aligned} \tag{34}$$

Thus we have two definitions for the generators E_{ij}, F_{ij} when $|i-j| \neq 1$ and we should check their consistency. The proof of this is inductive. We start with the case $|i-j| = 2$ where we have the desired consistency just using (26) and (27):

$$\langle E_{i,i+2}, f\rangle = \langle E_i E_{i+1} - p_i^{-1} E_{i+1} E_i, f\rangle = \delta_{m_{i,i+2}1}\delta_{m0}^{i,i+2}\delta_{n0} \tag{35a}$$

$$\langle F_{i+2,i}, f\rangle = \langle F_{i+1} F_i - p_i F_i F_{i+1}, f\rangle = \delta_{n_{i,i+2}1}\delta_{n0}^{i,i+2}\delta_{m0}. \tag{35b}$$

Then we suppose that we have proved consistency for E_{ij}, F_{ij} when $1 < |i-j| < s$ and then we shall prove for E_{ij}, F_{ij} for $|i-j| = s$. Namely, using this supposition and (26) and (27) we find that

$$\langle E_{ij}, f\rangle = \langle E_i E_{i+1,j} - p_{ij}^{-1} E_{i+1,j} E_i, f\rangle = \delta_{m_{ij}1}\delta_{m0}^{ij}\delta_{n0} \tag{36a}$$

$$\langle F_{ij}, f\rangle = \langle F_{i,j+1} F_j - p_{ij} F_j F_{i,j+1}, f\rangle = \delta_{n_{ij}1}\delta_{n0}^{ij}\delta_{m0}. \tag{36b}$$

For (36) we have used analogies of (26) and (27) for $E_i E_{i+1,j}$, $E_{i+1,j}E_i$, $F_{i,j+1}F_j$, $F_j F_{i,j+1}$.

4. Hopf algebra structure of the dual algebra

In this section we shall use the duality to derive the Hopf algebra structure of $\mathcal{U}_{uq}$. We start with the co-products in $\mathcal{U}_{uq}$, i.e. we shall repeatedly use the first of relations (14)

$$\langle Y, f\rangle = \langle \delta_{\mathcal{U}}(Y), f_1 \otimes f_2\rangle \tag{37}$$

for every splitting $f = f_1 f_2$. Thus we derive

$$\delta_{\mathcal{U}}(D_i) = D_i \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes D_i \tag{38a}$$

$$\delta_{\mathcal{U}}(H_i) = H_i \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes H_i, \delta_{\mathcal{U}}(K) = K \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes K. \tag{38b}$$

Then we try the following ansätze:

$$\delta_{\mathcal{U}}(E_i) = E_i \otimes \mathcal{P}_i + 1_{\mathcal{U}} \otimes E_i \tag{39a}$$

$$\delta_{\mathcal{U}}(F_i) = F_i \otimes \mathcal{Q}_i + 1_{\mathcal{U}} \otimes F_i. \tag{39b}$$

We take in (37) $f_1 = a_{i,i+1}$, $f_2 = (a_{11})^{k_1} \ldots (a_{nn})^{k_n}$ and using

$$f_1 f_2 = a_{i,i+1}(a_{11})^{k_1} \ldots (a_{nn})^{k_n} = A_i (a_{11})^{k_1} \ldots (a_{nn})^{k_n} a_{i,i+1} = A_i f_2 f_1 \tag{40a}$$

$$A_i = \left(\prod_{s=1}^{i-1} \left(\frac{q_{si}}{q_{s,i+1}}\right)^{k_s}\right) \left(\frac{u^2}{q_{i,i+1}}\right)^{k_i} \left(\frac{1}{q_{i,i+1}}\right)^{k_{i+1}} \left(\prod_{t=i+2}^{n} \left(\frac{q_{i+1,t}}{q_{it}}\right)^{k_t}\right) \tag{40b}$$

we obtain, on the one hand,

$$\langle E_i, f_1 f_2\rangle = A_i \langle E_i, f_2 f_1\rangle = A_i \tag{41}$$

while, on the other hand, using the ansatz (39a) we have

$$\langle E_i, f_1 f_2\rangle = \langle E_i, f_1\rangle\langle \mathcal{P}_i, f_2\rangle = \langle \mathcal{P}_i, f_2\rangle. \tag{42}$$

Comparing (41) with (42) we try

$$\mathcal{P}_i = \left(\prod_{s=1}^{i-1}\left(\frac{q_{si}}{q_{s,i+1}}\right)^{D_s}\right)\left(\frac{u^2}{q_{i,i+1}}\right)^{D_i}\left(\frac{1}{q_{i,i+1}}\right)^{D_{i+1}}\left(\prod_{t=i+2}^{n}\left(\frac{q_{i+1,t}}{q_{it}}\right)^{D_t}\right) \tag{43}$$

then we check that (39a) with this choice is consistent for all choices of f_1, f_2 in (37).

Analogously we proceed to obtain $\mathcal{Q}_i$: we take $f_1' = a_{i+1,i}$, $f_2 = (a_{11})^{k_1}\dots(a_{nn})^{k_n}$ to find

$$f_1' f_2 = a_{i+1,i}(a_{11})^{k_1}\dots(a_{nn})^{k_n} = u^{2(k_i - k_{i+1})} A_i^{-1} f_2 f_1' \tag{44}$$

and thus we have

$$\mathcal{Q}_i = \left(\prod_{s=1}^{i-1}\left(\frac{q_{s,i+1}}{q_{si}}\right)^{D_s}\right)(q_{i,i+1})^{D_i}\left(\frac{q_{i,i+1}}{u^2}\right)^{D_{i+1}}\left(\prod_{t=i+2}^{n}\left(\frac{q_{it}}{q_{i+1,t}}\right)^{D_t}\right) = u^{2H_i}\mathcal{P}_i^{-1}. \tag{45}$$

The co-products of the rest of the generators we obtain using (34) and the co-products of the generators E_i, F_i, e.g.

$$\delta_{\mathcal{U}}(E_{i,i+2}) = E_{i,i+2}\otimes\mathcal{P}_{i,i+2} + 1_{\mathcal{U}}\otimes E_{i,i+2} + (\lambda/u)E_{i+1}\otimes E_i\mathcal{P}_{i+1} \tag{46}$$

$$\mathcal{P}_{i,i+2} = \mathcal{P}_i\mathcal{P}_{i+1} = \left(\prod_{s=1}^{i-1}\left(\frac{q_{si}}{q_{s,i+2}}\right)^{D_s}\right)\left(\frac{u^2}{q_{i,i+2}}\right)^{D_i}\left(\frac{u^2}{q_{i,i+1}q_{i+1,i+2}}\right)^{D_{i+1}} \times\left(\frac{1}{q_{i,i+2}}\right)^{D_{i+2}}\left(\prod_{t=i+3}^{n}\left(\frac{q_{i+2,t}}{q_{it}}\right)^{D_t}\right) \tag{47}$$

$$\delta_{\mathcal{U}}(F_{i+2,i}) = F_{i+2,i}\otimes\mathcal{Q}_{i+2,i} + 1_{\mathcal{U}}\otimes F_{i+2,i} - u\lambda F_i\otimes F_{i+1}\mathcal{Q}_i \tag{48}$$

$$\mathcal{Q}_{i+2,i} = \mathcal{Q}_i\mathcal{Q}_{i+1} = u^{2(H_i+H_{i+1})}(\mathcal{P}_i\mathcal{P}_{i+1})^{-1} = u^{2H_{i,i+2}}\mathcal{P}_{i,i+2}^{-1} \tag{49}$$

$$H_{i,i+2} \equiv H_i + H_{i+1} = D_i - D_{i+2} \tag{50}$$

where we have used

$$\mathcal{P}_i E_j = \begin{cases} u^2 E_i\mathcal{P}_i & \text{for } i=j \\ g_{ij}^{-1}E_j\mathcal{P}_i & \text{for } i\neq j\end{cases} \qquad \mathcal{P}_i F_j = \begin{cases} u^{-2} F_i\mathcal{P}_i & \text{for } i=j \\ g_{ij}F_j\mathcal{P}_i & \text{for } i\neq j\end{cases} \tag{51a}$$

$$\mathcal{Q}_i E_j = \begin{cases} u^2 E_i\mathcal{Q}_i & \text{for } i=j \\ u^{2c_{ij}}g_{ij}E_j\mathcal{Q}_i & \text{for } i\neq j\end{cases} \qquad \mathcal{Q}_i F_j = \begin{cases} u^{-2} F_i\mathcal{Q}_i & \text{for } i=j \\ u^{-2c_{ij}}g_{ij}^{-1}F_j\mathcal{P}_i & \text{for } i\neq j\end{cases} \tag{51b}$$

The co-unit relations in $\mathcal{U}_{uq}$ are given by

$$\varepsilon_{\mathcal{U}}(Y) = 0 \qquad Y = D_i, E_{ij}, F_{ij}, K, H_i \tag{52}$$

which follows easily using (15), (20) and (29):

$$\varepsilon_{\mathcal{U}}(Y) = \langle Y, 1_{\mathcal{A}}\rangle = 0. \tag{53}$$

Finally the antipode map in $\mathcal{U} = \mathcal{U}_{uq}$ is given by

$$\gamma_{\mathcal{U}}(D_i) = -D_i \tag{54a}$$

$$\gamma_{\mathcal{U}}(H_i) = -H_i \qquad \gamma_{\mathcal{U}}(K) = -K \tag{54b}$$

$$\gamma_{\mathcal{U}}(E_i) = -E_i\mathcal{P}_i^{-1} \qquad \gamma_{\mathcal{U}}(F_i) = -F_i\mathcal{Q}_i^{-1}. \tag{54c}$$

This follows from (38), (39) and (52) with elementary application of one of the basic axioms of Hopf algebras [32]:

$$m \circ (\mathrm{id}_{\mathcal{U}} \otimes \gamma_{\mathcal{U}}) \circ \delta_{\mathcal{U}} = i \circ \varepsilon_{\mathcal{U}} \tag{55}$$

where both sides are maps $\mathcal{U} \to \mathcal{U}$, m is the usual product in the algebra: $m(Y \otimes Z) = YZ$, $Y, Z \in \mathcal{U}$ and i is the natural embedding of $\mathbb{C}$ into $\mathcal{U}$: $i(c) = c1_{\mathcal{U}}$, $c \in \mathbb{C}$. To obtain (54) we just apply both sides of (55) to D_i, H_i, K, E_i, F_i. For (54c) we also use $\gamma_{\mathcal{U}}(\mathcal{P}_i) = \mathcal{P}_i^{-1}$, $\gamma_{\mathcal{U}}(\mathcal{Q}_i) = \mathcal{Q}_i^{-1}$ which follows from (54a). The antipode map for the rest of the generators E_{ij}, F_{ij} we obtain using (34) and (54).

5. Drinfeld–Jimbo form of the dual algebra

In this section we show how to transform the algebra $\mathcal{U}_{uq}$ to a Drinfeld–Jimbo form. (It could be transformed also to the algebra given in [17] in terms only of the Chevalley generators.) We first note that if we set all parameters equal $q_{ij} = u$ for all i, j and make the change

$$E_i = X_i^+ u^{H_i/2} \qquad F_i = X_i^- u^{H_i/2} \tag{56}$$

then the generators H_i, $X_i^{\pm}$, $1 \leqslant i \leqslant n-1$ obey the commutation rules and Serre relations of the standard Drinfeld–Jimbo deformation $U_u(sl(n, \mathbb{C}))$.

Then we note that if $q_{ij} = u$ for all i, j then we have $\mathcal{P}_i = u^{H_i} = \mathcal{Q}_i$. Thus we are prompted to try for the analogue of the transformation (56) in the following multiparametric case:

$$E_i = X_i^+ \mathcal{P}_i^{1/2} \qquad F_i = X_i^- \mathcal{Q}_i^{1/2}. \tag{57}$$

Indeed we have

$$[H_i, X_j^+] = [H_i, E_j\mathcal{P}_i^{-1/2}] = c_{ij}X_j^+ \tag{58a}$$

$$[H_i, X_j^-] = [H_i, F_j\mathcal{Q}_i^{-1/2}] = -c_{ij}X_j^- \tag{58b}$$

$$[X_i^+, X_i^-] = [E_i\mathcal{P}_i^{-1/2}, F_i\mathcal{Q}_i^{-1/2}] = (uE_iF_i - u^{-1}F_iE_i)u^{-H_i} = \lambda^{-1}(u^{H_i} - u^{-H_i}) \equiv [H_i]_u \tag{58c}$$

where we have used (57), (31b–d), (43), (45) and (51):

$$[X_i^+, X_j^-] = [E_i\mathcal{P}_i^{-1/2}, F_j\mathcal{Q}_j^{-1/2}] = E_iF_j\left(\frac{\mathcal{P}_j}{g_{ij}\mathcal{P}_i}\right)^{1/2} u^{-H_j} - F_jE_i\left(\frac{\mathcal{P}_j}{g_{ji}\mathcal{P}_i}\right)^{1/2} u^{-H_j}u^{\delta_{j,i\pm1}}$$

$$= F_jE_i\left(\frac{\mathcal{P}_j}{g_{ji}\mathcal{P}_i}\right)^{1/2} u^{-H_j}(g_{ij}^{1/2}g_{ji}^{1/2} - u^{\delta_{j,i\pm1}}) = 0 \qquad \text{for } i \neq j \tag{59}$$

where we have used (57), (43), (45), (51) and

$$g_{ij}g_{ji} = \begin{cases} u^2 & \text{for } j = i \pm 1 \\ 1 & \text{otherwise} \end{cases} \tag{60}$$

next we have

$$\begin{aligned}(X_i^+)^2 X_{i\pm1}^+ &- [2]_u X_i^+ X_{i\pm1}^+ X_i^+ + X_{i\pm1}^+ (X_i^+)^2 \\ &= u^{-1} g_{i,i\pm1} E_i^2 E_{i\pm1} - [2]_u E_i E_{i\pm1} E_i + u^{-1} g_{i\pm1,i} E_{i\pm1} E_i^2 = 0\end{aligned} \tag{61}$$

where we have used (57), (51) the facts that $g_{i,i\pm1}/u = p_i^{\pm}$, $g_{i\pm1,i}/u = (p_i^{\pm})^{-1}$ and (32a);

$$\begin{aligned}X_i^+ X_j^+ &= E_i \mathcal{P}_i^{-1/2} E_j \mathcal{P}_j^{-1/2} = g_{ij}^{1/2} E_i E_j \mathcal{P}_i^{-1/2} \mathcal{P}_j^{-1/2} = g_{ij}^{-1/2} E_j E_i \mathcal{P}_i^{-1/2} \mathcal{P}_j^{-1/2} \\ &= g_{ij}^{-1/2} g_{ji}^{-1/2} E_j \mathcal{P}_j^{-1/2} E_i \mathcal{P}_i^{-1/2} = X_j^+ X_i^+ \qquad i < j-1\end{aligned} \tag{62}$$

where we have used (57), (51), (28e) and (60). Formulae (31a), (58), (59), (61), (62) and the analogies of (61), (62) for the '$-$' sign are the defining relations of the one-parameter deformation $U_u(sl(n,\mathbb{C}))$ in term of the Chevalley generators H_i, $X_i^{\pm}$, $i = 1,\ldots,n-1$.

Thus as a *commutation algebra* we have $\mathcal{U}_{uq} \cong U_u(sl(n,\mathbb{C})) \otimes U_u(\mathcal{Z})$, where $U_u(\mathcal{Z})$ is spanned by $K, u^{\pm K/2}$. This splitting is also preserved by the co-unit and the antipode, cf (52) and (54b) for the generators H_i and K, while for $X_i^{\pm}$ we have

$$\varepsilon_{\mathcal{U}}(X_i^+) = \varepsilon_{\mathcal{U}}(E_i)\varepsilon_{\mathcal{U}}(\mathcal{P}_i^{-1/2}) = 0 \qquad \varepsilon_{\mathcal{U}}(X_i^-) = \varepsilon_{\mathcal{U}}(F_i)\varepsilon_{\mathcal{U}}(\mathcal{Q}_i^{-1/2}) = 0 \tag{63a}$$

$$\gamma_{\mathcal{U}}(X_i^+) = \gamma_{\mathcal{U}}(\mathcal{P}_i^{-1/2})\gamma_{\mathcal{U}}(E_i) = -\mathcal{P}_i^{1/2} E_i \mathcal{P}_i^{-1} = -u E_i \mathcal{P}_i^{-1/2} = -u X_i^+ \tag{63b}$$

$$\gamma_{\mathcal{U}}(X_i^-) = \gamma_{\mathcal{U}}(\mathcal{Q}_i^{-1/2})\gamma_{\mathcal{U}}(F_i) = -\mathcal{Q}_i^{1/2} F_i \mathcal{P}_i^{-1} = -u^{-1} F_i \mathcal{Q}_i^{-1/2} = -u^{-1} X_i^- \tag{63c}$$

where we have used (51). The splitting is also preserved by the co-products of H_i, K, cf (38b).

However, for the co-products of the Chevalley generators $X_i^{\pm}$ we have

$$\delta_{\mathcal{U}}(X_i^+) = \delta_{\mathcal{U}}(E_i)\delta_{\mathcal{U}}(\mathcal{P}_i^{-1/2}) = (E_i \otimes \mathcal{P}_i + 1_{\mathcal{U}} \otimes E_i)(\mathcal{P}_i^{-1/2} \otimes \mathcal{P}_i^{-1/2}) = X_i^+ \otimes \mathcal{P}_i^{1/2} + \mathcal{P}_i^{-1/2} \otimes X_i^+ \tag{64a}$$

$$\begin{aligned}\delta_{\mathcal{U}}(X_i^-) &= \delta_{\mathcal{U}}(F_i)\delta_{\mathcal{U}}(\mathcal{Q}_i^{-1/2}) = (F_i \otimes \mathcal{Q}_i + 1_{\mathcal{U}} \otimes F_i)(\mathcal{Q}_i^{-1/2} \otimes \mathcal{Q}_i^{-1/2}) \\ &= X_i^- \otimes \mathcal{Q}_i^{1/2} + \mathcal{Q}_i^{-1/2} \otimes X_i^- .\end{aligned} \tag{64b}$$

Thus, as a co-algebra $\mathcal{U}_{uq}$ cannot be split as above and furthermore it depends on all parameters. Only if we set $q_{ij} = u$ for all i, j then $\mathcal{P}_i = u^{H_i} = \mathcal{Q}_i$ and (64) become the standard co-products of the Chevalley generators $X_i^{\pm}$ of $U_u(sl(n,\mathbb{C}))$.

Finally, a remark is in order on the non-degeneracy of the pairing (18) used for the duality. We know that if we set $q_{ij} = u = 1$ for all i, j then we recover the classical duality between $GL(n,\mathbb{C})$ and $U(sl(n,\mathbb{C})) \otimes U(\mathcal{Z})$ with the same pairing (18). Thus the pairing (18) is not degenerate in the classical case. Suppose, now that it is degenerate in the multiparameter deformed case considered here. This would mean that there is some relation:

$$\langle v, a \rangle = 0 \tag{65}$$

where $v \in \mathcal{U}_{uq}$ is some fixed non-zero element and $a \in GL_{uq}(n)$ is arbitrary (or a is fixed and v arbitrary), and furthermore this relation becomes trivial, i.e. $0 = 0$, when $q_{ij} = u = 1$ for all i, j. The latter is possible only if the element v is becoming zero itself in the classical limit (because of non-degeneracy). This would mean that $v = \sum_i k_i v_i$ is a polynomial consisting of basis monomials v_i with non-zero classical limit with coefficients k_i which all vanish in the classical case. However, since (65) is valid for any a, this means that in the deformed case we have an infinite number of equations for a finite number of unknowns with at least one non-trivial solution given by the coefficients k_i. This would mean that there are some additional relations in $\mathcal{U}_{uq}$ besides (58), (59), (61), (62) which relations would become trivial in the classical limit. From our analysis of the dual algebra it seems plausible that such unnatural relations do not exist.

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